The result of computing the values of the temperature at the points $M_i^{(s)}$ of the i-th ray with coordinates $r_i^{(s)} = d_i + 0.1s$, $s = 1, 2, \ldots, \theta_i = i\tilde{h}$ is presented in Table 1.

The data in Table 1 yield a representation of the temperature distribution in the internal points of the plate area. They are obtained upon partitioning the interval (0, π/4) into eight parts (i = 0, 1, ..., 8) with a division spacing of $\tilde{h} = \pi/32$.

NOTATION

L_1, L_2, plate contours; r, Θ, dimensionless polar coordinates, r_ν, ε_ν, m_ν, a_ν, ν = 1, 2, contour parameters; h(r, Θ), plate thickness; H, P, given functions; T_ν, ν = 1, 2, value of the temperature on the L_ν contour; T, function of the temperature; λ, heat-conduction coefficient: τ, Kirchhoff variable; Φ, Ψ, known functions; α, parameter playing the part of the eigennumber; $\tilde{\Theta}$, period of the solution of the problem; n, number parts into which the interval is divided; h, division spacing; Θ_i, point of division; Y_i, an approximate value of the function y(Θ) at the division point; μ, parameter; $F(\Theta_i)$ known function; μ_k, roots of the characteristic equation; $f_i^{(k)}$, $\varphi_1^{(k)}$, $\varphi_2^{(k)}$, functions of the radius r; $C_i^{(k)}$, $D_i^{(k)}$, constants of integration; τ_i, a function of the radius r at the i-th ray; X_k, Z_k, parameters determined from the boundary conditions of the problem; $M_i^{(s)}$, a point of the i-th ray; and $r_i^{(s)}$, Θ_i, coordinates of a point on the i-th ray.

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APPROXIMATE ANALYTICAL SOLUTION OF LINEAR HEAT-CONDUCTION PROBLEMS IN REGIONS WITH NONCANONICAL BOUNDARIES

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We present a method for solving linear heat-conduction problems in regions bounded by a noncanonical contour. The method is based on extending the noncanonical contour to a contour imbedded in the grid of classical coordinate systems.

The use of various modifications of the method of partial regions (see, for example, [1]) broadens the possibility of analytically solving heat-conduction problems. The main ingredient in the application of these methods is the requirement of a canonical contour bounding the computational region (it must be formed by the intersection of orthogonal coordinate surfaces of classical coordinate systems [2]).

In the present paper we offer an approximate analytical solution of linear heat-conduction problems in regions bounded by a noncanonical contour.

In connection with fields described by the Laplace equation, our method for the solution of a problem can be represented as follows: 1) a contour of complex profile bounding the computational region is extended to a contour of canonical form; 2) on the extended part of the contour a boundary condition of the second kind

$$\lambda \frac{\partial T}{\partial n}\bigg|_s = q(s),$$

is introduced, where q(s) is an unknown thermal flow distribution function on the "extended" boundary s; 3) the function q(s) may be replaced by a piecewise-constant representation q_i, i= 1,2,...,M; 4) a solution of a field problem constructed by one of the analytical

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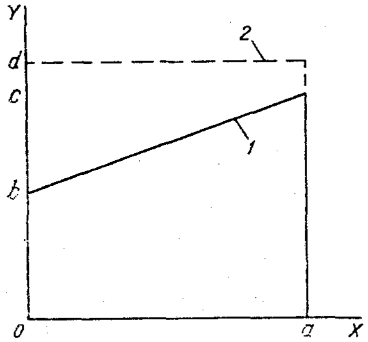

Fig. 1. Shape of the computational region: line 1 is a boundary of the initial region; line 2 is an "added-on" canonical boundary; a, b, c, and d are node coordinates of corresponding regions.

methods described in [2, 3] over the whole "expanded" region will be a parametric function of the unknown thermal flows q_i; 5) a set q_i is sought which will satisfy the boundary conditions of the initial problem at the M nodes of a collocation [4] located on the initial (noncanonical) countour. We consider a specific example. Assume that we need to solve the Laplace equation in the region outlined by the continuous curve in Fig. 1:

$$\nabla^2 T = 0$$

and that we need to satisfy the set of boundary conditions

$$T|_{y=b+\frac{c-b}{a}x,\, x\in(0,a)} = U(x), \tag{2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0,\, y\in(0,b)} = 0, \tag{3}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=a,\, y\in(0,c)} = 0, \tag{4}$$

$$T|_{y=0,\, x\in(0,a)} = 0. \tag{5}$$

We proceed to solve an auxiliary problem in which we add on a contour, bounding the computational region, of canonical form (the dashed line in Fig. 1). On the extended part of the contour (boundary y = d) we introduce the boundary condition

$$\lambda \left.\frac{\partial T}{\partial y}\right|_s = q(x)$$

on the boundary x = 0, y ∈ (0, d) we have condition (3), and on the boundary x = a, y ∈ (0, d) we have condition (4). On the boundary y = 0, x ∈ (0, a) condition (5) stays unchanged.

We replace the function q(x) by the piecewise-smooth representation $q(x) = q_i$, x ∈ ((i − 1)a/M, ic/M), i = 1, 2, ..., M. The solution of the auxiliary problem, obtained by the method of separation of variables [3], has the form

$$T(x,\ y,\ q_i,\ i=1,\ 2,\ \ldots,\ M) = \sum_{k=1}^{\infty} A_k \operatorname{sh}(\omega_k y)\cos(\omega_k x) + A_0 y, \tag{6}$$

where

$$A_k = \frac{4\sin\left(\frac{a}{2M}\omega_k\right)}{a\omega_k\lambda \operatorname{ch}(\omega_k d)} \sum_{i=1}^{M} q_i \cos\left(\frac{a}{2M}(2i-1)\omega_k\right),$$

$$A_0 = \frac{1}{M\lambda}\sum_{i=1}^{M} q_i, \quad \omega_k = \frac{k\pi}{a}.$$

After this, we reduce the problem to that of finding the set of values q_i which provide the temperatures $U(x_i)$, i = 1, 2, ..., M, at collocation points distributed along the boundary of the initial contour $y_i = b + (c - b)x_i/a$, $x_i \in (0, a)$.

Using the principle of superposition of thermal fields, valid for linear heat-conduction problems [3], we can write

$$U_i = U(x_i) = \sum_{j=1}^{M} a_{ij} q_j, \quad i = 1,\ 2,\ \ldots,\ M. \tag{7}$$

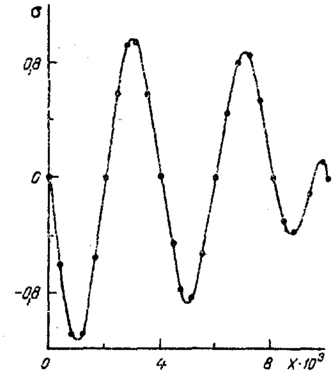

Fig. 2. Distribution of relative error σ (%) for a synthesis of condition (2) along the boundary $y = b + (c - b)x/a$ as a solution of the auxiliary problem with M subdivisions of the contour boundary.

We can obtain the coefficients a_{ij} appearing in Eq. (7) upon making appropriate analytical transformations of formula (6) or from the relation

$$a_{ij} = T(x_i,\ y_i = b + (c - b)\,x_i/a,\quad q_1 = q_2 = \ldots = q_{j-1} = q_{j+1} = \ldots = q_M = 0,\ q_j = 1). \tag{8}$$

From the physical point of view the coefficient a_{ij} characterizes the value of the temperature at the i-th node of the collocation (x_i, $y_i = b + (c - b)x_i/a$) per unit thermal flow ($q_j = 1$) introduced at the j-th interval of the added-on contour.

The unknowns q_i are found by solving the linear system of algebraic equations (8) by the method of Gauss [5]. Substitution of q_i into the relation (6) yields an approximate analytical solution of the initial problem (1)-(5).

A numerical solution of our problem was carried out on the BÉSM-6 computer for the following values of the parameters: $a = 10^{-2}$ m, $b = 0.8 \cdot 10^{-2}$ m, $c = 10^{-2}$ m, $d = 10^{-2}$ m, $\lambda = 10$ W/(m•deg), $x_1 = 0$, $x_2 = 2 \cdot 10^{-3}$ m, $x_3 = 4 \cdot 10^{-3}$ m, $x_4 = 6 \cdot 10^{-3}$ m, $x_5 = 8 \cdot 10^{-3}$ m, $x_6 = 10^{-2}$ m, $M = 6$, $U_1 = U_2 = U_3 = U_4 = U_5 = U_6 = 200°C$.

Since the solution obtained satisfies the Laplace equation (1) exactly in the computational region and satisfies the boundary conditions (3)-(5) on the boundaries ($x = 0$, $y \in (0, b)$, ($x = a$, $y \in (0, c)$, $y = 0$, $x \in (0, a)$), the maximum relative error in the computed temperature at an arbitrary point of the region does not exceed the relative error of the synthesis of condition (2) on the boundary $y = b + (c - b)x/a$, $x \in (0, a)$. Figure 2 shows the distribution of the relative error in the synthesis of the boundary condition (2) of the initial problem (1)-(5) by means of the flows q_i, $i = 1, 2, \ldots, 6$. It is evident from the figure that the maximum relative error in the computed temperature field is at most 1.2%. Naturally, an increase in the number of collocation nodes increases the accuracy of the solution. Thus, when $M = 10$ the relative error does not exceed 0.4%; moreover, the time to calculate the temperature at 400 points amounts to around 20 sec. Calculation of this problem by a grid method using a KSI program on the BÉSM-6 computer requires a machine time two orders of magnitude greater to achieve the same degree of accuracy.

NOTATION

λ, thermal conductivity coefficient; n, unit vector in the direction of the exterior normal to the boundary of the region; T, temperature distribution function; M, number of subdivisions of the contour boundary.

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